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The new composite implicit iterative process with errors for common fixed points of a finite family of strictly pseudocontractive mappings ☆

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Abstract

Convergence theorems for approximation of common fixed points of strictly pseudocontractive mappings of Browder–Petryshyn type are proved in Banach spaces using a new composite implicit iteration scheme with errors. The results presented in this paper generalize and improve the corresponding results of M.O. Osilike [M.O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl. 294 (2004) 73–81].

Keywords: Strictly pseudocontractive mappings; Implicit iteration process with errors; Common fixed points

1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) = \{f \in E^*\colon \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$, $\forall x \in E$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. If E^* is strictly

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convex, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j.

Definition 1.1. Let K be a closed subset of real Banach space E and $T : K \to K$ be a mapping. T is said to be *semi-compact*, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ ($n \to \infty$), then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$.

Definition 1.2. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *nonexpansive* if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in D(T). \tag{1.1}$$

Definition 1.3. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Browder and Petryshyn [1], if for all $x, y \in D(T)$, there exist $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\big\langle Tx - Ty, j(x - y)\big\rangle \leqslant \|x - y\|^2 - k\big\|x - y - (Tx - Ty)\big\|^2. \tag{1.2}$$

If I denotes the identity operator, then (1.2) can be written in the form

$$\big\langle (I - T)x - (I - T)y, j(x - y)\big\rangle \geqslant k\big\|(I - T)x - (I - T)y\big\|^2. \tag{1.3}$$

It is easy to know that every strictly pseudocontractive mapping is L-Lipschitzian and continuous. Indeed, it follows from (1.3) that

$$k\big\|(x - y) - (Tx - Ty)\big\|^2 \leqslant \big\|(x - y) - (Tx - Ty)\big\| \cdot \big\|j(x - y)\big\|,$$
$$k\big(\|Tx - Ty\| - \|x - y\|\big) \leqslant k\big\|(x - y) - (Tx - Ty)\big\| \leqslant \|x - y\|,$$

i.e.,

$$\|Tx - Ty\| \leqslant L\|x - y\|, \quad \text{where } L = \frac{k + 1}{k}.$$

The class of strictly pseudocontractive mappings has been studied by several authors (see, for example, [1,3–5,7–11]).

Let K be a nonempty convex subset of E, and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of K. In [12], Xu and Ori introduced the following implicit iteration process. For any $x_0 \in K$ and $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{cases} x_1 = (1 - \alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 = (1 - \alpha_2)x_1 + \alpha_2 T_2 x_2, \\ \quad \vdots \\ x_N = (1 - \alpha_N)x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} = (1 - \alpha_{N+1})x_N + \alpha_{N+1} T_1 x_{N+1}, \\ \quad \vdots \end{cases}$$

which can be written in the following compact form as follows:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_n T_n x_n, \quad \forall n \geqslant 1, \tag{1.4}$$

where $T_n = T_{n \,(\mathrm{mod}\, N)}$.

Using this iteration process, they proved the following convergence theorem for nonexpansive mappings in Hilbert spaces.

Theorem XO. [12] *Let H be a Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N : K \to K$ be N nonexpansive mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $x_0 \in K$ and $\{\alpha_n\}$ be a sequence in* $(0, 1)$ *with* $\lim_{n\to\infty}(1 - \alpha_n) = 0$. *Then the sequence $\{x_n\}$ defined by* (1.4) *converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Recently, M.O. Osilike [6] extended their results from the nonexpansive mappings to strictly pseudocontractive mappings. By this iteration process, he proved the following convergence theorems in Hilbert and Banach spaces.

Theorem MO1. [6] *Let H be a Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N : K \to K$ be N strictly pseudocontractive mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $x_0 \in K$ and $\{\alpha_n\}$ be a sequence in* $(0, 1)$ *with* $\lim_{n\to\infty}(1 - \alpha_n) = 0$. *Then the sequence $\{x_n\}$ defined by* (1.4) *converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Theorem MO2. [6] *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : K \to K$ be N strictly pseudocontractive mappings such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $x_0 \in K$ and $\{\alpha_n\}$ be a sequence in* $(0, 1)$ *satisfying the conditions*:

(i) $0 < \alpha_n < 1$,
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$,
(iii) $\sum_{n=1}^\infty \alpha_n^2 < \infty$.

Then the sequence $\{x_n\}$ defined by (1.4) *converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

In this paper, we introduce a new composite implicit iteration process as follows:

$$\begin{cases} x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_n y_n + \gamma_n u_n, & n \geqslant 1, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T_n x_n + \delta_n v_n, & n \geqslant 1, \end{cases} \tag{1.5}$$

where $T_n = T_{n \pmod N}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leqslant 1$ and $\beta_n + \delta_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K and x_0 is a given point.

Especially, if $\{\alpha_n\}$, $\{\gamma_n\}$ be two sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ be a bounded sequence in K and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_n x_n + \gamma_n u_n, \quad \forall n \geqslant 1. \tag{1.6}$$

The purpose of this paper is to study the convergence of implicit iterative sequence $\{x_n\}$ defined by (1.5) and (1.6) to a common fixed point for a finite family of strictly pseudocontractive mappings of Browder–Petryshyn type in an arbitrary real Banach spaces. The results presented in this paper generalized and extend the corresponding results of M.O. Osilike [6], even in the case of $\beta_n = \delta_n = 0$, $\forall n \geqslant 1$ or $N = 1$ are also new. Moreover, in this paper the methods of proof of main results are also different from that of Osilike.

In order to prove the main results of this paper, we need the following lemmas:

Lemma 1.1. [2] *Let E be a real Banach space and let J be the normalized duality mapping. Then for any given $x, y \in E$, we have*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 1.2. [7] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following condition*:

$$a_{n+1} \leqslant (1+b_n)a_n + c_n, \quad \forall n \geqslant n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then:

(1) *The limit $\lim_{n\to\infty} a_n$ exists.*
(2) *In addition, if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ ($n \to \infty$).*

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \ldots, T_N\}: K \to K$ be N strictly pseudocontractive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \ldots, T_N\}$). Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in $[0,1]$ satisfying $\alpha_n + \gamma_n \leqslant 1$ and $\beta_n + \delta_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;
(iv) $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;
(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (1.5)*, then the following conclusions hold*:

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all $p \in F$*;
(ii) $\liminf_{n\to\infty} \|x_n - T_n x_n\| = 0$.

Proof. Since each $T_i : K \to K$, $i \in I = \{1, 2, \ldots, N\}$, be strictly pseudocontractive, then we have $\forall x, y \in K$, there exist constants $k_i \in (0,1)$ and $L_i \geqslant 1$ such that

$$\langle T_i x - T_i y, j(x-y)\rangle \leqslant \|x-y\|^2 - k_i \big\|x - T_i x - (y - T_i y)\big\|^2, \quad \forall i \in I,$$

and

$$\|T_i x - T_i y\| \leqslant L_i \|x-y\|, \quad \forall i \in I.$$

Let $k = \min_{1\leqslant i\leqslant N}\{k_i\}$ and $L = \max_{1\leqslant i\leqslant N}\{L_i\}$, then

$$\langle T_i x - T_i y, j(x-y)\rangle \leqslant \|x-y\|^2 - k \big\|x - T_i x - (y - T_i y)\big\|^2, \quad \forall i \in I, \tag{2.1}$$

and

$$\|T_i x - T_i y\| \leqslant L\|x - y\|, \quad \forall i \in I. \tag{2.2}$$

Let $p \in F$, it follows from (1.5), (2.1), (2.2) and Lemma 1.1 that

$$\begin{aligned}
\|x_n - p\|^2 &= \big\|(1-\alpha_n-\gamma_n)(x_{n-1}-p) + \alpha_n(T_n y_n - p) + \gamma_n(u_n - p)\big\|^2 \\
&\leqslant (1-\alpha_n-\gamma_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n\big\langle T_n y_n - p, j(x_n-p)\big\rangle \\
&\quad + 2\gamma_n\big\langle u_n - p, j(x_n-p)\big\rangle \\
&= (1-\alpha_n-\gamma_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n\big\langle T_n y_n - T_n x_n, j(x_n-p)\big\rangle \\
&\quad + 2\alpha_n\big\langle T_n x_n - p, j(x_n-p)\big\rangle + 2\gamma_n\big\langle u_n - p, j(x_n-p)\big\rangle \\
&\leqslant (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n\|T_n y_n - T_n x_n\|\cdot\|x_n-p\| + 2\alpha_n\|x_n-p\|^2 \\
&\quad - 2\alpha_n k\|x_n - T_n x_n\|^2 + 2\gamma_n\|u_n-p\|\cdot\|x_n-p\| \\
&\leqslant (1-\alpha_n)^2\|x_{n-1}-p\|^2 + 2\alpha_n L\|y_n - x_n\|\cdot\|x_n-p\| + 2\alpha_n\|x_n-p\|^2 \\
&\quad - 2\alpha_n k\|x_n - T_n x_n\|^2 + 2\gamma_n\|u_n-p\|\cdot\|x_n-p\|.
\end{aligned} \tag{2.3}$$

From (1.5) and (2.2), we also have that

$$\begin{aligned}
\|y_n - x_n\| &= \big\|\beta_n(T_n x_n - x_n) + \delta_n(v_n - x_n)\big\| \\
&\leqslant \beta_n\|T_n x_n - x_n\| + \delta_n\|v_n - x_n\| \\
&\leqslant \beta_n(L+1)\|x_n-p\| + \delta_n\|x_n-p\| + \delta_n\|v_n-p\| \\
&= \big(\beta_n(L+1)+\delta_n\big)\|x_n-p\| + \delta_n\|v_n-p\|.
\end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.3), we obtain that

$$\begin{aligned}
\|x_n - p\|^2 &\leqslant (1-\alpha_n)^2\|x_{n-1}-p\|^2 + \big[2\alpha_n\beta_n L(L+1) + 2\alpha_n\delta_n L + 2\alpha_n\big]\|x_n-p\|^2 \\
&\quad + 2\gamma_n\|u_n-p\|\cdot\|x_n-p\| + 2\alpha_n\delta_n L\|v_n-p\|\cdot\|x_n-p\| \\
&\quad - 2\alpha_n k\|x_n - T_n x_n\|^2.
\end{aligned} \tag{2.5}$$

Setting $M_1 = \max\{\sup\{\|u_n-p\|^2\colon n \geqslant 1\}, \sup\{\|v_n-p\|^2\colon n \geqslant 1\}\}$, and noticing that $2\|u_n - p\|\cdot\|x_n-p\| \leqslant \|u_n-p\|^2 + \|x_n-p\|^2$ and $2\|v_n-p\|\cdot\|x_n-p\| \leqslant \|v_n-p\|^2 + \|x_n-p\|^2$, it follows from (2.5) that

$$\begin{aligned}
\|x_n - p\|^2 &\leqslant (1-\alpha_n)^2\|x_{n-1}-p\|^2 + \big[2\alpha_n\beta_n L(L+1) + 2\alpha_n\delta_n L + 2\alpha_n\big]\|x_n-p\|^2 \\
&\quad + \gamma_n\big(\|u_n-p\|^2 + \|x_n-p\|^2\big) + \alpha_n\delta_n L\big(\|v_n-p\|^2 + \|x_n-p\|^2\big) \\
&\quad - 2\alpha_n k\|x_n - T_n x_n\|^2 \\
&= (1-\alpha_n)^2\|x_{n-1}-p\|^2 \\
&\quad + \big[2\alpha_n\beta_n L(L+1) + 3\alpha_n\delta_n L + 2\alpha_n + \gamma_n\big]\|x_n-p\|^2 \\
&\quad + \gamma_n\|u_n-p\|^2 + \alpha_n\delta_n L\|v_n-p\|^2 - 2\alpha_n k\|x_n - T_n x_n\|^2 \\
&\leqslant (1-\alpha_n)^2\|x_{n-1}-p\|^2 \\
&\quad + \big[2\alpha_n\beta_n L(L+1) + 3\alpha_n\delta_n L + 2\alpha_n + \gamma_n\big]\|x_n-p\|^2 \\
&\quad + \gamma_n M_1 + \alpha_n\delta_n L M_1 - 2\alpha_n k\|x_n - T_n x_n\|^2.
\end{aligned} \tag{2.6}$$

Transposing and simplifying above inequality, we have

$$
\begin{aligned}
\|x_n - p\|^2 &\leqslant \left(\frac{(1-\alpha_n)^2}{1-2\alpha_n\beta_n L(L+1) - 3\alpha_n\delta_n L - 2\alpha_n - \gamma_n}\right)\|x_{n-1}-p\|^2 \\
&\quad + \frac{\gamma_n M_1 + \alpha_n\delta_n L M_1}{1-2\alpha_n\beta_n L(L+1) - 3\alpha_n\delta_n L - 2\alpha_n - \gamma_n} \\
&\quad - \left(\frac{2\alpha_n k}{1-2\alpha_n\beta_n L(L+1) - 3\alpha_n\delta_n L - 2\alpha_n - \gamma_n}\right)\|x_n - T_n x_n\|^2 \\
&\leqslant \left(1 + \frac{\alpha_n^2 + 2\alpha_n\beta_n L(L+1) + 3\alpha_n\delta_n L + \gamma_n}{1-2\alpha_n\beta_n L(L+1) - 3\alpha_n\delta_n L - 2\alpha_n - \gamma_n}\right)\|x_{n-1}-p\|^2 \\
&\quad + \frac{\gamma_n M_1 + \alpha_n\delta_n L M_1}{1-2\alpha_n\beta_n L(L+1) - 3\alpha_n\delta_n L - 2\alpha_n - \gamma_n} - 2\alpha_n k\|x_n - T_n x_n\|^2 \\
&= \left(1 + \frac{\mu_n}{1-\sigma_n}\right)\|x_{n-1}-p\|^2 + \frac{\gamma_n M_1 + \alpha_n\delta_n L M_1}{1-\sigma_n} \\
&\quad - 2\alpha_n k\|x_n - T_n x_n\|^2,
\end{aligned} \tag{2.7}
$$

where $\mu_n = \alpha_n^2 + 2\alpha_n\beta_n L(L+1) + 3\alpha_n\delta_n L + \gamma_n$ and $\sigma_n = 2\alpha_n\beta_n L(L+1) + 3\alpha_n\delta_n L + 2\alpha_n + \gamma_n$. It follows from the conditions (ii)–(v) that

$$\sigma_n = 2\alpha_n\beta_n L(L+1) + 3\alpha_n\delta_n L + 2\alpha_n + \gamma_n \to 0 \quad (n \to \infty),$$

therefore there exists a natural number n_0 such that $1 - \sigma_n \geqslant \frac{1}{2}$ for any $n \geqslant n_0$. Hence, from (2.7) we have

$$
\begin{aligned}
\|x_n - p\|^2 &\leqslant (1+2\mu_n)\|x_{n-1}-p\|^2 + 2(\gamma_n M_1 + \alpha_n\delta_n L M_1) - 2\alpha_n k\|x_n - T_n x_n\|^2 \\
&= (1+b_n)\|x_{n-1}-p\|^2 + c_n - 2\alpha_n k\|x_n - T_n x_n\|^2, \quad \forall n \geqslant n_0,
\end{aligned} \tag{2.8}
$$

where $b_n = 2\mu_n$ and $c_n = 2(\gamma_n M_1 + \alpha_n\delta_n L M_1)$. From the conditions (ii)–(v) it is easy to see that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Thus using (2.8) and Lemma 1.2 we have limit $\lim_{n\to\infty}\|x_n - p\|^2$ exists, and so limit $\lim_{n\to\infty}\|x_n - p\|$ exists (since $\|x_n - p\| \geqslant 0$).

Since $\lim_{n\to\infty}\|x_n - p\|$ exists, then $\{x_n\}$ is bounded, hence there exists constant $M_2 > 0$ such that $\|x_n - p\|^2 \leqslant M_2, \forall n \geqslant 1$. It also follows from (2.8) that

$$
\begin{aligned}
2\alpha_n k\|x_n - T_n x_n\|^2 &\leqslant \|x_{n-1}-p\|^2 - \|x_n - p\|^2 + b_n\|x_{n-1}-p\|^2 + c_n \\
&\leqslant \|x_{n-1}-p\|^2 - \|x_n - p\|^2 + b_n M_2 + c_n, \quad \forall n \geqslant n_0.
\end{aligned}
$$

Thus

$$2k\sum_{j=n_0+1}^{\infty}\alpha_j\|x_j - T_j x_j\|^2 \leqslant \|x_{n_0} - p\|^2 + M_2\sum_{j=n_0+1}^{\infty} b_j + \sum_{j=n_0+1}^{\infty} c_j,$$

and hence

$$2k\sum_{n=1}^{\infty}\alpha_n\|x_n - T_n x_n\|^2 \leqslant \|x_n - p\|^2 + M_2\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n. \tag{2.9}$$

By virtue of the $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, it follows from (2.9) that

$$\sum_{n=1}^{\infty}\alpha_n\|x_n - T_n x_n\|^2 < \infty.$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, then we must have

$$\liminf_{n \to \infty} \|x_n - T_n x_n\| = 0.$$

This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \ldots, T_N\}: K \to K$ be N strictly pseudocontractive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \ldots, T_N\}$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ are two real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ be a bounded sequence in K satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (1.6)*, then the following conclusions hold*:

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F$;
(ii) $\liminf_{n\to\infty} \|x_n - T_n x_n\| = 0$.

Proof. Taking $\beta_n = \delta_n = 0$, $\forall n \geqslant 1$, in Theorem 2.1, then the conclusion of Corollary 2.2 can be obtained from Theorem 2.1 immediately. This completes the proof of Corollary 2.2. □

Theorem 2.3. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \ldots, T_N\}: K \to K$ be N strictly pseudocontractive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \ldots, T_N\}$). Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leqslant 1$ and $\beta_n + \delta_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;
(iv) $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;
(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (1.5)*, then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings family $\{T_i\}_{i=1}^{N}$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.10}$$

Proof. The necessity of condition (2.10) is obvious.

Next we prove the sufficiency of Theorem 2.3. For any given $p \in F$, it follows from (2.8) in Theorem 2.1 that

$$\|x_n - p\|^2 \leqslant (1 + b_n)\|x_{n-1} - p\|^2 + c_n, \quad \forall n \geqslant n_0, \tag{2.11}$$

where sequences $\{b_n\}$ and $\{c_n\}$ satisfying $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Hence, we have

$$\big[d(x_n, F)\big]^2 \leqslant (1 + b_n)\big[d(x_{n-1}, F)\big]^2 + c_n, \quad \forall n \geqslant n_0. \tag{2.12}$$

It follows from (2.12) and Lemma 1.2 that the limit $\lim_{n\to\infty}[d(x_n, F)]^2$ exists, further, limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (2.10), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum_{n=1}^{\infty} b_n < \infty$, $1 + t \leqslant \exp\{t\}$ for all $t > 0$, and (2.11), therefore we have

$$\|x_n - p\|^2 \leqslant \exp\{b_n\}\|x_{n-1} - p\|^2 + c_n, \quad n \geqslant n_0. \tag{2.13}$$

Hence, for any positive integers n, m, $n \geqslant n_0$, from (2.13) we have

$$\begin{aligned}
\|x_{n+m} - p\|^2 &\leqslant \exp\{b_{n+m}\}\|x_{n+m-1} - p\|^2 + c_{n+m} \\
&\leqslant \exp\{b_{n+m}\}\big[\exp\{b_{n+m-1}\}\|x_{n+m-2} - p\|^2 + c_{n+m-1}\big] + c_{n+m} \\
&= \exp\{b_{n+m} + b_{n+m-1}\}\|x_{n+m-2} - p\|^2 + \exp\{b_{n+m}\}c_{n+m-1} + c_{n+m} \\
&\leqslant \cdots \\
&\leqslant \exp\left\{\sum_{i=n+1}^{n+m} b_i\right\}\|x_n - p\|^2 + \exp\left\{\sum_{i=n+2}^{n+m} b_i\right\}\sum_{i=n+1}^{n+m} c_i \\
&\leqslant W\|x_n - p\|^2 + W\sum_{i=n+1}^{\infty} c_i,
\end{aligned}$$

where $W = \exp\{\sum_{n=1}^{\infty} b_n\} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, for any given $\epsilon > 0$, there exists a positive integer $n_1 \geqslant n_0$ such that

$$\big[d(x_n, F)\big]^2 < \frac{\epsilon^2}{8(W+1)}, \qquad \sum_{i=n+1}^{\infty} c_i < \frac{\epsilon^2}{4W}, \quad \forall n \geqslant n_1.$$

Therefore there exists $p_1 \in F$ such that

$$\|x_n - p_1\|^2 < \frac{\epsilon^2}{4(W+1)}, \quad \forall n \geqslant n_1.$$

Consequently, for any $n \geqslant n_1$ and for all $m \geqslant 1$ we have

$$\begin{aligned}
\|x_{n+m} - x_n\|^2 &\leqslant 2\big(\|x_{n+m} - p_1\|^2 + \|x_n - p_1\|^2\big) \\
&\leqslant 2(1 + W)\|x_n - p_1\|^2 + 2W\sum_{i=n+1}^{\infty} c_i \\
&< 2 \cdot \frac{\epsilon^2}{4(W+1)}(1 + W) + 2W \cdot \frac{\epsilon^2}{4W} \\
&= \epsilon^2,
\end{aligned}$$

i.e.,

$$\|x_{n+m} - x_n\| < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $x_n \to x^* \in K$. Observe that if $T : K \to K$ is strictly pseudocontractive and $\{p_n\}_{n=1}^{\infty}$ is a sequence in $F(T)$ which converges strongly to some p, then

$$\begin{aligned}\|p - Tp\| &\leqslant \|p - p_n\| + \|p_n - Tp\| \\ &= \|p - p_n\| + \|Tp_n - Tp\| \\ &\leqslant (1 + L)\|p - p_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Thus $p \in F(T)$, so that $F(T)$ is closed. It follows that $F(T_i)$ is closed for all $i \in I$, so that F is closed. Since

$$\lim_{n\to\infty} d(x_n, F) = 0,$$

we must have that $x^* \in F$. This completes the proof of Theorem 2.3. □

Corollary 2.4. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \dots, T_N\} : K \to K$ be N strictly pseudocontractive mappings with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \dots, T_N\}$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in* $[0, 1]$ *satisfying $\alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ be a bounded sequence in K satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (1.6)*, then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings family $\{T_i\}_{i=1}^{N}$ if and only if the condition* (2.10) *is satisfied.*

Proof. Taking $\beta_n = \delta_n = 0$, $\forall n \geqslant 1$, in Theorem 2.3, then the conclusion of Corollary 2.4 can be obtained from Theorem 2.3 immediately. This completes the proof of Corollary 2.4. □

In the case of $N = 1$, (1.5) becomes the implicit iteration process as follows:

$$\begin{cases} x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T y_n + \gamma_n u_n, & n \geqslant 1, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T x_n + \delta_n v_n, & n \geqslant 1. \end{cases} \tag{2.14}$$

The conclusion of Theorems 2.1 and 2.3 are still valid for the iteration process (2.14). Furthermore, we have the following result:

Theorem 2.5. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a semi-compact strictly pseudocontractive mappings with $F(T) = \{x \in K\colon Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in* $[0, 1]$ *satisfying $\alpha_n + \gamma_n \leqslant 1$ and $\beta_n + \delta_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;
(iv) $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;
(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (2.14)*, then the sequence $\{x_n\}$ convergence strongly to a fixed point of T.*

Proof. By Theorem 2.1 we known that

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0,$$

then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0. \tag{2.15}$$

By the semi-compactness of T, there must exist a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{i\to\infty} x_{n_{k_i}} = p_0.$$

It follows from (2.15) that $p_0 = Tp_0$, hence $p_0 \in F(T)$. Since $\lim_{n\to\infty} \|x_n - p_0\|$ exists, then

$$\lim_{n\to\infty} x_n = p_0.$$

This completes the proof of Theorem 2.5. □

Corollary 2.6. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a semi-compact strictly pseudocontractive mappings with $F(T) = \{x \in K\colon Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leqslant 1$ for all $n \geqslant 1$, $\{u_n\}$ be a bounded sequence in K satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n Tx_n + \gamma_n u_n, \quad n \geqslant 1. \tag{2.16}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Taking $\beta_n = \delta_n = 0$, $\forall n \geqslant 1$, in Theorem 2.5, then the conclusion of Corollary 2.6 can be obtained from Theorem 2.5 immediately. This completes the proof of Corollary 2.6. □

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